Solitons with Fusion and Fission Properties in the (2+1)-Dimensional Modified Dispersive Water-Wave System

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In this paper, by means of the general projective Riccati equation method (PREM), the variable separation solutions of the (2+1)-dimensional modified dispersive water-wave system are obtained. By further studying, we find that these variable separation solutions, which seem independent, actually depend on each other. Based on the special variable separation solution and choosing suitable functions p and q, soliton fusion and fission phenomena among peakons, compactons, dromions and semifoldons are firstly investigated. – PACS numbers: 05.45.Yv, 02.30.Jr, 02.03Ik

Key words: General Projective Riccati Equation Method; (2+1)-Dimensional MDWW System; Soliton Fusion; Soliton Fission.

1. Introduction

It is well-known that the interactions of solitons in (1+1)-dimensional nonlinear models are usually considered to be elastic. That means there is no exchange of any physical quantities like the energy and the momentum among interacting solitons. That is, the amplitude, velocity and wave shape of a soliton do not undergo any change after the nonlinear interaction. However, for some (1+1)-dimensional models, two or more solitons may fuse into one soliton at a special time while sometimes one soliton may fission into two or more solitons at another special time [1]. These phenomena are often called soliton fusion and soliton fission, respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems, such as organic membranes and macromolecular material [2], plasma physics, nuclear physics and hydrodynamics [3]. Recently, Zhang et al. [4] and Lin et al. [5] studied the evolutions of soliton solutions for two (1+1)-dimensional partial differential equations (PDEs) with time and revealed the soliton fission and soliton fusion phenomena. In a similar way, via Hirota's direct method, Wang et al. [6] further discussed two (1+1)-dimensional models, i. e., the Burgers equation and the Sharma-Tasso-Olver equation, and also found the soliton fission and soliton fusion phenomena. Moreover, Fang and Zheng [7] revealed special soliton fission and fusion phenomena in the (2+1)-dimensional Broer-Kaup-Kupershmidt (BKK) system. However, to the best of our knowledge, very few reports on soliton fusion and fission behaviors in higher-dimensional nonlinear physical models have been found. The fusion and fission phenomena among higher-dimensional coherent localized structures, such as peakons, compactons, dromions and semifoldons, are hardly discussed. Motivated by this reason, we will report and discuss these phenomena in the following well-known (2+1)-dimensional modified dispersive water-wave (MDWW) system:

$$\begin{aligned} &u_{ty} + u_{xxy} - 2v_{xx} - (u^2)_{xy} = 0, \\ &v_t - v_{xx} - 2(uv)_x = 0, \end{aligned} \tag{1}$$

which was used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth. It may be derived from the inner parameter-dependent symmetry constraint of the celebrated Kadomtsev-Petviashvili (KP) equation [8]. It is worth mentioning that this system has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics. Therefore, a good understanding of more solutions of the MDWW system (1) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model in a harbor and coastal design. Abundant propagating localized excitations were derived by Tang et al. [9] with the help of a Painlevé-Bäcklund transformation and a multilinear variable separation approach. Abundant non-propagating and propagating solitons were also revealed in [10].

Recently, many effective and direct methods, such as the extended tanh-function method (ETM) based on the mapping method [11], the projective Riccati equation method (PREM) [12] and the Jacobian-function method [13], were developed to derive travelling wave solutions of the nonlinear evolutional equations. Among these direct methods, the ETM based on the mapping method has been an alternative method to obtain variable separation solutions for the (2+1)-dimensional BKK system [14], MDWW system [10], dispersive long wave (DLW) equation [15], generalized Broer-Kaup (BK) system [16], and so on. More recently, we also derived the variable separation solutions of a large type of nonlinear evolutional equations by means of the ETM [17]. Now a natural and important issue is that whether the variable separation solutions based on the former ETM can be derived by other direct methods. The crucial question is how to obtain solutions with certain arbitrary functions.

In this paper, we successfully generalize the PREM to obtain variable separation solutions of the (2+1)-dimensional MDWW equation. By further studying, we find that these variable separation solutions obtained by the PREM, which seem independent, actually depend on each other. We can see this fact from variable separation solutions of the (2+1)-dimensional MDWW system, which will be discussed in detail as an example.

2. Review of the General Projective Riccati Equation Method

The basic idea of the general PREM is that for a given nonlinear partial differential equation (NPDE) with independent variables $x(=x_0=t,x_1,x_2,\ldots,x_m)$ and dependent variables u

$$L(u,u_t,u_{x_i},u_{x_ix_j},\cdots)=0, \tag{2}$$

where L is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives.

Step 1: One assumes that (2) possesses the following ansatz:

Type I

$$u=a_0(x)+\sum_{i=1}^{l} f^{i-1}[w(x)] \cdot\{a_i(x)f[w(x)]+b_i(x)g[w(x)]\}, \tag{3}$$

where $a_0=a_0(x)$, $a_i=a_i(x)$, $b_i=b_i(x)$ $(i=1,\cdots,l)$ and $w=w(x)$ are all arbitrary functions of indicated variables. $f(w)$ and $g(w)$ satisfy

$$\begin{aligned} &f'(w)=\varepsilon f(w)g(w),\\ &g'(w)=R+\varepsilon g^2(w)-rf(w), \quad \varepsilon=\pm1, \end{aligned} \tag{4}$$

$$\begin{aligned} &g^2(w)=-\varepsilon[R-2rf(w)+\frac{r^2+\mu}{R}f^2(w)],\\ &R\neq 0,\ \mu=\pm1, \end{aligned} \tag{5}$$

where R and r are two constants, and $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}w}$. When $R=r=0$ in (4) and (5), we seek solutions of (2) in the following form:

Type II

$$u=a_0(x)+\sum_{i=1}^{l} a_i(x)g^i[w(x)], \tag{6}$$

where $g(w)$ satisfies

$$g'(w)=g^2(w). \tag{7}$$

The parameter l is determined by balancing the highest-order derivative terms with the nonlinear terms in (2).

Step 2: Substituting (3) along with (4) and (5) [or (6) along with (7)] into (2) yields a set of polynomials for f^ig^j $(i=0,1,\cdots;j=0,1)$. Eliminating all the coefficients of the powers of f^ig^j, yields a series of partial differential equations, from which the parameters a_0, a_i, b_i $(i=1,\cdots,l)$ and w are explicitly determined.

Step 3: We know from [12, 18–20] that (4) and (5) admit the following solutions:

Case 1. When $\varepsilon=-1$, $\mu=-1$, $R\neq 0$,

$$\begin{aligned} &f_1(w)=\frac{R\,\mathrm{sech}(\sqrt{R}w)}{r\,\mathrm{sech}(\sqrt{R}w)+1},\\ &g_1(w)=\frac{\sqrt{R}\tanh(\sqrt{R}w)}{r\,\mathrm{sech}(\sqrt{R}w)+1}. \end{aligned} \tag{8}$$

Case 2. When $\varepsilon=-1$, $\mu=1$, $R\neq 0$,

$$f_2(w)=\frac{R\operatorname{csch}(\sqrt{R}w)}{r\operatorname{csch}(\sqrt{R}w)+1},\quad g_2(w)=\frac{\sqrt{R}\coth(\sqrt{R}w)}{r\operatorname{csch}(\sqrt{R}w)+1}. \tag{9}$$

Case 3. When $\varepsilon=1$, $\mu=-1$, $R\neq 0$,

$$f_3(w)=\frac{R\sec(\sqrt{R}w)}{r\sec(\sqrt{R}w)+1},\quad g_3(w)=\frac{\sqrt{R}\tan(\sqrt{R}w)}{r\sec(\sqrt{R}w)+1}. \tag{10}$$

$$f_4(w)=\frac{R\csc(\sqrt{R}w)}{r\csc(\sqrt{R}w)+1},\quad g_4(w)=-\frac{\sqrt{R}\cot(\sqrt{R}w)}{r\csc(\sqrt{R}w)+1}. \tag{11}$$

Case 4. When $R=r=0$,

$$f_5(w)=\frac{C}{w}=C\varepsilon g_5(w),\quad g_5(w)=\frac{1}{\varepsilon w}, \tag{12}$$

where C is a constant. Inserting the parameters a_0, a_i, b_i $(i=1,\cdots,l)$ and w obtained in step 2 into (3) along with (4) and (5) [or (6) along with (7)], we can obtain variable separation solutions of the NPDE (2).

Remark 1. When $\varepsilon=-1$, $R=1$, $\mu=\mu/k$, (2) becomes a projective Riccati equation [12]. When $\varepsilon=-1$, $R=1$, (2) becomes the projective Riccati equation, by means of which a (2+1)-dimensional simplified generalized Broer-Kaup (SGBK) system was studied in [18].

Remark 2. The PREM is firstly generalized to derive variable separation solutions of the nonlinear evolutional equation in this paper. Actually, these solutions in (8)–(12), which seem independent, depend on each other if they are applied to find variable separation solutions for nonlinear soliton systems. This fact can be concluded from the variable separation solutions of the (2+1)-dimensional MDWW equation, which will be discussed in detail in Section 3. In these solutions (8)–(12), only the solution (12) is essentially effective, while other solutions related to tan, cot, tanh and coth functions are special cases of (12).

3. Variable Separation Solutions for the (2+1)-Dimensional MDWW System

To solve the (2+1)-dimensional MDWW system, first, let us make a transformation for (1): $v=u_y$. Substituting the transformation into (1) yields

$$u_{ty}-2(u_xu)_y-u_{xxy}=0. \tag{13}$$

Along with the general PREM, according to step 2 in Section 2, by balancing the higher-order derivative term with the nonlinear term in (13), we get $l=1$ in (3). Therefore we suppose that (13) has the following formal solution with $R\neq 0$:

$$u(x,y,t)=a_0(x,y,t)+a_1(x,y,t)f(w)+b_1(x,y,t)g(w), \tag{14}$$

where f and g satisfy (4) and (5) with (8)–(11), and $w\equiv w(x,y,t)$. Inserting (14) with (4) and (5) into (13), choosing the variable separation ansatz

$$w=p(x,t)+q(y), \tag{15}$$

and eliminating all the coefficients of polynomials of f^ig^j $(i=0\sim 4;\ j=0,1)$, one gets a set of partial differential equations. It is very difficult to solve these prolix and complicated differential equations. Fortunately, by careful analysis and calculation, we derive the special solutions

$$a_0=-\frac{p_{xx}-p_t}{2p_x},\quad a_1=\frac{p_x}{2}\sqrt{\frac{-\varepsilon(r^2+\mu)}{R}},\quad b_1=-\frac{p_x}{2\varepsilon}. \tag{16}$$

Therefore, the variable separation solutions of the (2+1)-dimensional MDWW system read:

Case 1. $\varepsilon=-1$, $\mu=-1$,

$$\begin{aligned}u_1&=-\frac{p_{xx}-p_t}{2p_x}+\frac{p_x}{2}\sqrt{\frac{r^2-1}{R}}\frac{R\operatorname{sech}[\sqrt{R}(p+q)]}{r\operatorname{sech}[\sqrt{R}(p+q)]+1}\\&\quad+\frac{p_x}{2}\frac{\sqrt{R}\tanh[\sqrt{R}(p+q)]}{r\operatorname{sech}[\sqrt{R}(p+q)]+1}\\&\equiv-\frac{p_{xx}-p_t}{2p_x}+\frac{p_x}{2}\sqrt{R}\tanh[\sqrt{R}\frac{(p+q)+\varphi_1}{2}],\end{aligned} \tag{17}$$

$$v_1=u_{1y}=\frac{R}{4}p_xq_y\operatorname{sech}^2[\sqrt{R}\frac{(p+q)+\varphi_1}{2}]. \tag{18}$$

In (17), we use the relation $\frac{\sqrt{r^2-1}\operatorname{sech}(\theta)+\tanh(\theta)}{r\operatorname{sech}(\theta)+1}\equiv\tanh(\frac{\theta+\varphi_1}{2})$, where $\varphi_1=\tanh^{-1}(\frac{\sqrt{r^2-1}}{r})$.

Case 2. $\varepsilon = -1$, $\mu = 1$,

$$u_2 = -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{2}\sqrt{\frac{r^2+1}{R}}\frac{R\mathrm{csch}[\sqrt{R}(p+q)]}{r\mathrm{csch}[\sqrt{R}(p+q)]+1} + \frac{p_x}{2}\frac{\sqrt{R}\coth[\sqrt{R}(p+q)]}{r\mathrm{csch}[\sqrt{R}(p+q)]+1} \equiv -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{2}\sqrt{R}\coth[\sqrt{R}\frac{(p+q)+\varphi_2}{2}], \quad (19)$$

$$v_2 = u_{2y} = -\frac{R}{4}p_x q_y \mathrm{csch}^2[\sqrt{R}\frac{(p+q)+\varphi_1}{2}]. \quad (20)$$

In (19), we use the relation $\frac{\sqrt{r^2+1}\mathrm{csch}(\theta)+\coth(\theta)}{r\mathrm{csch}(\theta)+1} \equiv \coth(\frac{\theta+\varphi_2}{2})$, where $\varphi_2 = \tanh^{-1}(\frac{r}{\sqrt{r^2+1}})$.

Case 3. $\varepsilon = 1$, $\mu = -1$,

$$u_3 = -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{2}\sqrt{\frac{1-r^2}{R}}\frac{R\sec[\sqrt{R}(p+q)]}{r\sec[\sqrt{R}(p+q)]+1} - \frac{p_x}{2}\frac{\sqrt{R}\tan[\sqrt{R}(p+q)]}{r\sec[\sqrt{R}(p+q)]+1} \equiv -\frac{p_{xx}-p_t}{2p_x} - \frac{p_x}{2}\sqrt{R}\tan[\sqrt{R}\frac{(p+q)-\varphi_3}{2}], \quad (21)$$

$$v_3 = u_{3y} = -\frac{R}{4}p_x q_y \sec^2[\sqrt{R}\frac{(p+q)-\varphi_3}{2}]. \quad (22)$$

In (21), we use the relation $\frac{\sqrt{1-r^2}\sec(\theta)-\tan(\theta)}{r\sec(\theta)+1} \equiv -\tan(\frac{\theta-\varphi_3}{2})$, where $\varphi_3 = \tan^{-1}(\frac{\sqrt{1-r^2}}{r})$.

$$u_4 = -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{2}\sqrt{\frac{1-r^2}{R}}\frac{R\csc[\sqrt{R}(p+q)]}{r\csc[\sqrt{R}(p+q)]+1} + \frac{p_x}{2}\frac{\sqrt{R}\cot[\sqrt{R}(p+q)]}{r\csc[\sqrt{R}(p+q)]+1} \equiv -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{2}\sqrt{R}\cot[\sqrt{R}\frac{(p+q)+\varphi_4}{2}], \quad (23)$$

$$v_4 = u_{4y} = -\frac{R}{4}p_x q_y \csc^2[\sqrt{R}\frac{(p+q)+\varphi_4}{2}]. \quad (24)$$

In (23), we use the relation $\frac{\sqrt{1-r^2}\csc(\theta)+\cot(\theta)}{r\csc(\theta)+1} \equiv \cot(\frac{\theta+\varphi_4}{2})$, where $\varphi_4 = \tan^{-1}(\frac{r}{\sqrt{1-r^2}})$.

According to the method mentioned above in Section 2, when $R = r = 0$ and assuming that (13) has solutions of the form $u(x,y,t) = a_0(x,y,t) + a_1(x,y,t)g(w)$, then we obtain the rational solutions

$$u_5 = -\frac{p_{xx}-p_t}{2p_x} + \frac{p_x}{p+q}, \quad (25)$$

$$v_5 = u_{5y} = -\frac{p_x q_y}{(p+q)^2}, \quad (26)$$

where p and q are arbitrary functions of $\{x,t\}$ and $\{y\}$, respectively.

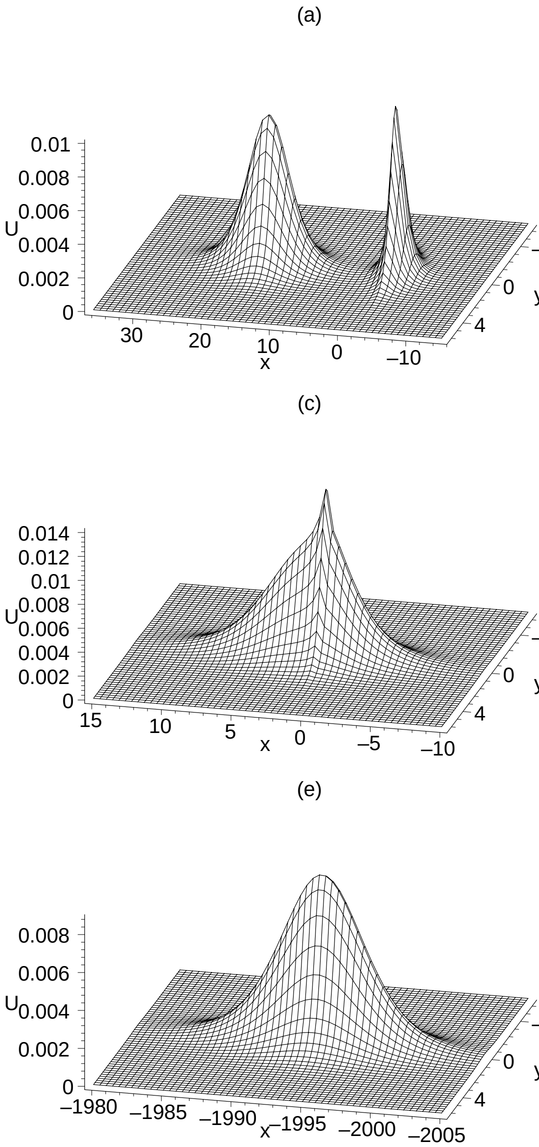
By careful analysis, we find that when re-defining $p = \exp\{-\sqrt{R}p\}$, $q = \exp\{\sqrt{R}(q+\varphi_1)\}$ in solutions (25) and (26), solutions (17) and (18) can be obtained. Similarly, if taking $p = -\exp\{-\sqrt{R}p\}$, $q = \exp\{\sqrt{R}(q+\varphi_2)\}$ in solutions (25) and (26), solutions (19) and (20) can be recovered. When considering $p = \exp\{-\mathrm{i}\sqrt{R}p\}$, $q = \exp\{\mathrm{i}\sqrt{R}(q+\varphi_1)\}$ in solutions (25) and (26), solutions (21) and (22) can be obtained. If letting $p = -\exp\{-\mathrm{i}\sqrt{R}p\}$, $q = \exp\{\mathrm{i}\sqrt{R}(q+\varphi_1)\}$ in solutions (25) and (26), solutions (23) and (24) can be recovered. Therefore, only solutions (25) and (26) are essentially effective.

4. Soliton Fusion and Fission among Peakons, Compactons, Dromions and Semifoldons

In this section, we will discuss some interesting localized structures for the quantity U:

$$U \equiv v_5 = -\frac{p_x q_y}{(p+q)^2}, \quad (27)$$

where $p = p(x,t)$ and $q = q(y)$. From (27), we know that for general choices of p and q there may some singularities for the quantity U. However, when the arbitrary functions p and q are chosen appropriately to avoid the singularities, there may exist abundant excitations for U. All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons, can be re-derived by the quantity U expressed by (27). Moreover, if p or q is considered to be a periodic function or a solution of a chaos system like the Lorenz chaos system, then solitons possess periodic or chaotic behaviors. It is well-known that there are some lower-dimensional stochastic fractal functions, which may be used to construct higher-dimensional stochastic fractal dromion and lump excitations by the quantity U expressed by (27). Since these similar situations have been widely discussed in literatures [9, 14 – 17], the related plots are neglected in our present paper. From the above brief discussions, one may conclude that the field U may possess some novel properties that have not been revealed until now. Recently, it has been reported both theoretically and experimentally that fission and fusion phenomena can happen for (1+1)-dimensional solitons or solitary waves [6]. Now we pay



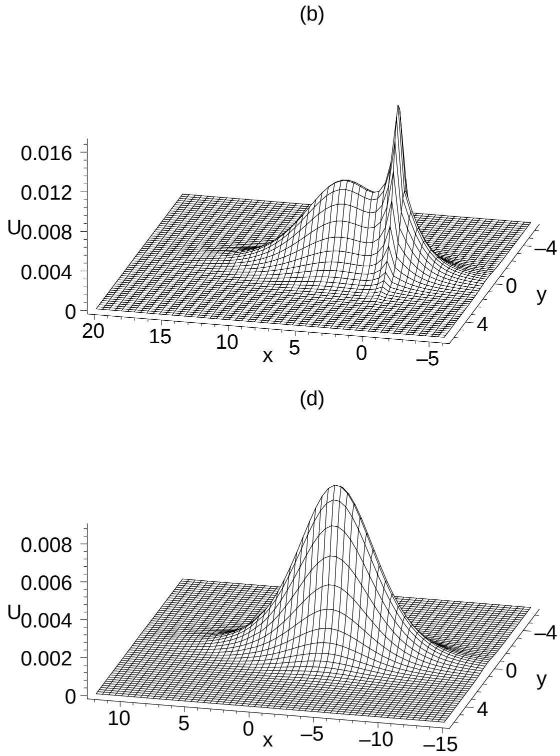


Fig. 1. Dromion and peakon fuse into a single soliton: time evolutional profiles for the quantity U (27) with conditions (28) and (29) at different times: (a) $t=-5$; (b) $t=1$; (c) $t=2$; (d) $t=5$; (e) $t=1000$.

our attention to the soliton fusion and fission phenomena among peakons, compactons, dromions and semifoldons in the (2+1)-dimensional MDWW system.

4.1. Fusion Phenomenon between Dromion and Peakon

When choosing the arbitrary functions p and q to be

$$p = 28 + \exp(0.5x + t) + \begin{cases} 5\exp(x - t/2), & \text{if } x - t/2 \le 0, \\ -5\exp(-x + t/2), & \text{if } x - t/2 > 0, \end{cases} \tag{28}$$

$$q = -2\tanh(y), \tag{29}$$

we can obtain a new kind of fusion solitary wave solution for the MDWW system. From Fig. 1, one can find that the left dromion moves along the negative x-axis, and the right peakon travels along the positive x-axis, then they interact and fuse into one soliton finally. The fused single soliton stably moves along the negative x-axis for subsequent time as we run the program for rather long time ($t = 10^3$).

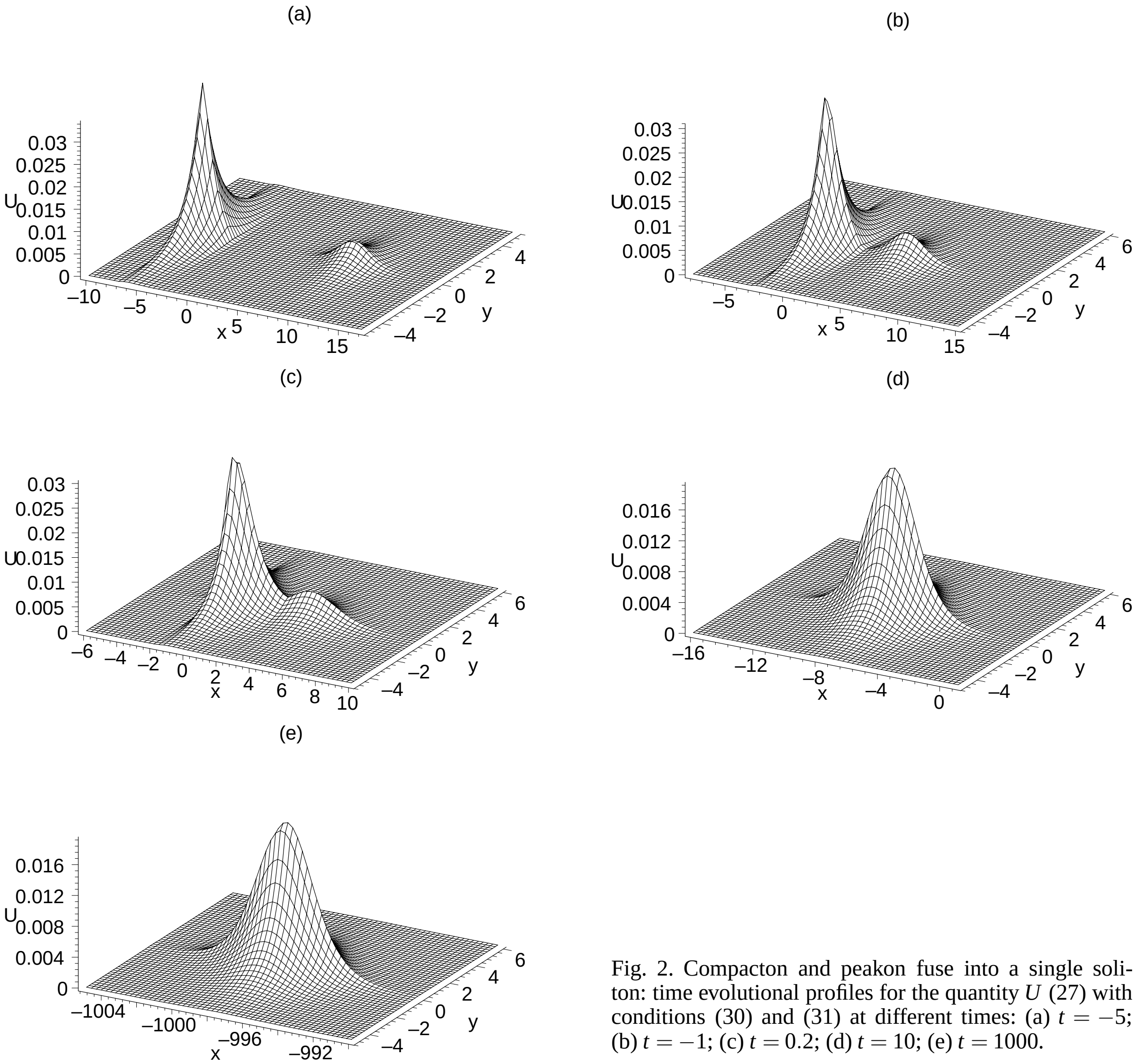


Fig. 2. Compacton and peakon fuse into a single soliton: time evolutional profiles for the quantity U (27) with conditions (30) and (31) at different times: (a) $t = -5$; (b) $t = -1$; (c) $t = 0.2$; (d) $t = 10$; (e) $t = 1000$.

4.2. Fusion Phenomenon between Compacton and Peakon

Similarly, if p and q are taken as

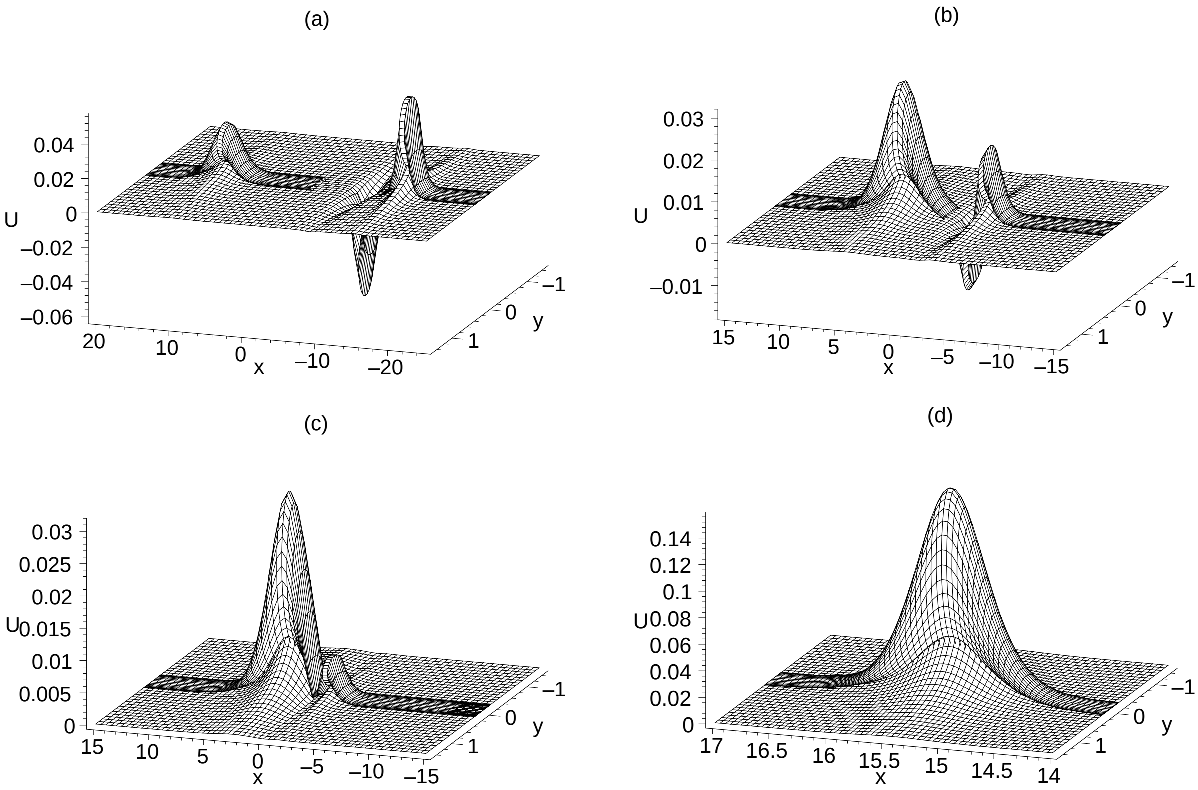
$$p = 12 + \exp(x+t) + \begin{cases} 0, & \text{if } x - t \le -\frac{5\pi}{8}, \\ 16\sin(0.8x - 0.8t) + 16, & \\ \qquad \text{if } -\frac{5\pi}{8} < x - t \le \frac{5\pi}{8}, & \\ 32, & \text{if } x - t > \frac{5\pi}{8}, \end{cases} \tag{30}$$

$$q = \begin{cases} -\exp(y), & \text{if } y < 0, \\ \exp(-y), & \text{if } y \ge 0, \end{cases} \tag{31}$$

then the fusion phenomenon between compacton and peakon can be observed, as depicted in Figure 2. The big peakon travels along the positive x-axis, and the small compacton moves along the negative x-axis, and ultimately they fuse into a single soliton running along the negative x-axis stably even while we run the program for rather long time ($t = 10^3$).

4.3. Fusion Phenomenon among Bell-Like Semifoldons

Besides the fusion phenomena between usual localized coherent structures discussed in Sections 4.1. and 4.2., we also find some novel fusion phenomenon



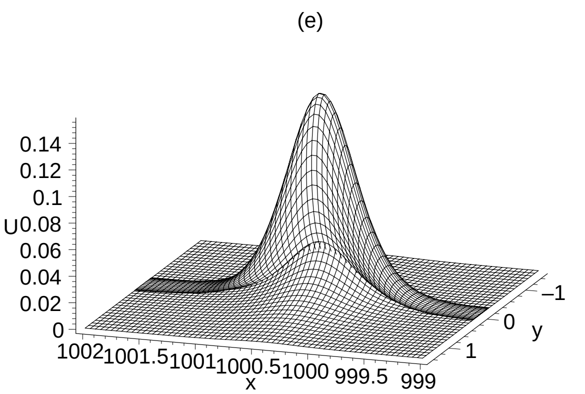


Fig. 3. Two bell-like semifoldons and one anti-bell-like semifoldon merge into a single bell-like semifoldon: time evolutional profiles for the quantity U (27) with conditions (32) and (33) at different times: (a) $t=-8$; (b) $t=-2$; (c) $t=-0.5$; (d) $t=15$; (e) $t=1000$.

among multi-valued (folded) localized excitations, i. e. bell-like semifoldons. If p and q possess the following forms:

$$p=-8-[\exp(5x-5t)+0.8\exp(2x-3t)+\exp(2x-4t)][1+\exp(2x+3t)]^{-2}, \tag{32}$$

$$q_y=\operatorname{sech}^2(\zeta),\quad x=\zeta-1.5\tanh(\zeta),\quad q=\int^{\zeta} q_y y_\zeta \mathrm{d}\zeta, \tag{33}$$

one can observe the intriguing fusion phenomenon, plotted in Figure 3. Different from the soliton fusion behaviors in Figs. 1 and 2, three semifoldons (two bell-like semifoldons and one anti-bell-like semifoldon) merge into a single bell-like semifoldon moving stably along the positive x-axis in the end.

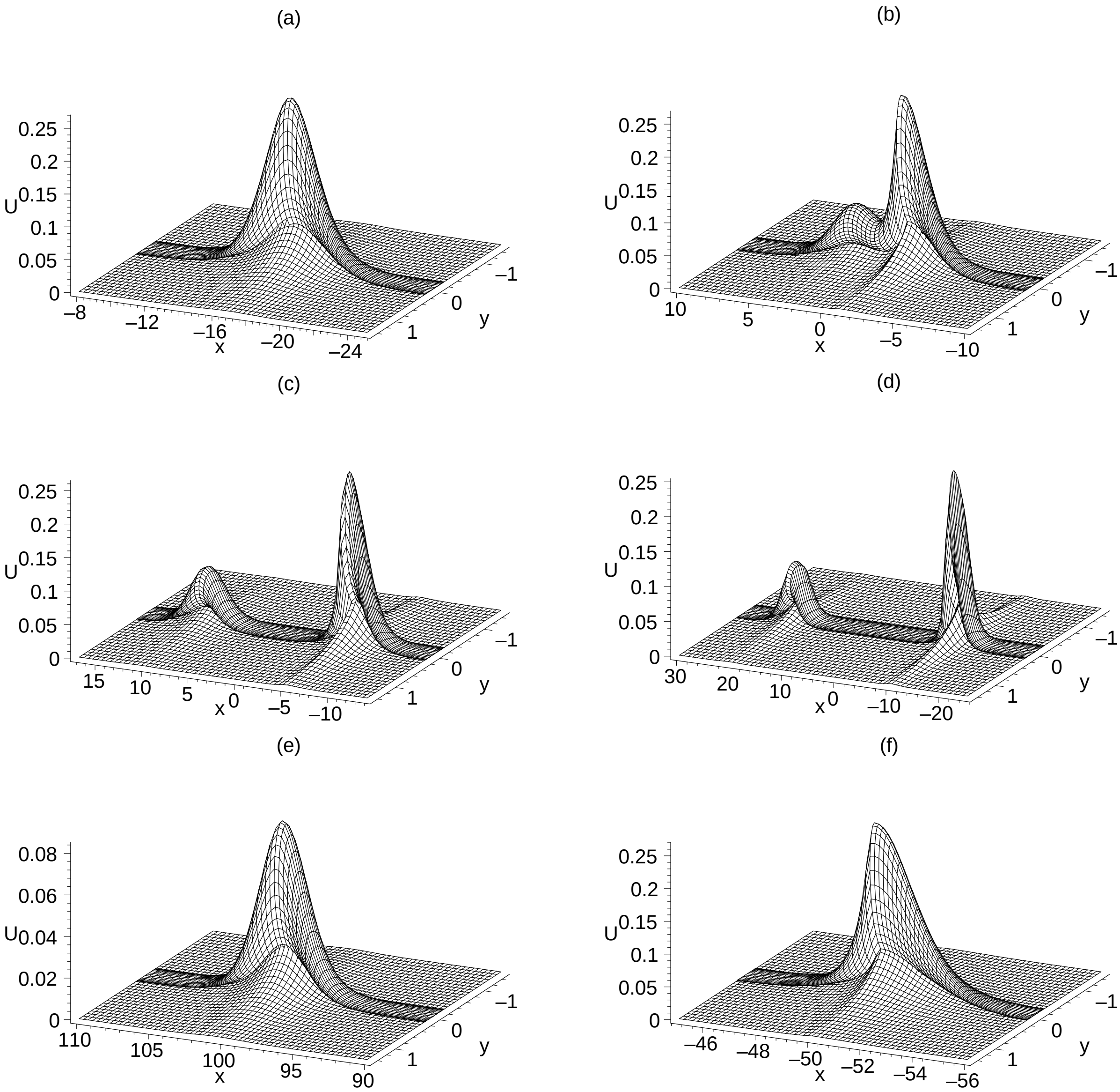


Fig. 4. Single bell-like semifoldon fissions fuse into two bell-like semifoldons: time evolutional profiles for the quantity U (27) with conditions (34) and (35) at different times: (a) $t=-8$; (b) $t=1$; (c) $t=5$; (d) $t=10$. (e), (f) The stable left and right travelling bell-like semifoldons at time $t=50$.

4.4. Fission Phenomenon of Single Bell-Like Semifoldon

If considering the arbitrary functions p and q as

$$p = 1 + 2\exp(x-2t) + \begin{cases} \exp(x+t), & \text{if } x+t \le 0, \\ -\exp(-x-t/2), & \text{if } x+t > 0, \end{cases} \tag{34}$$

$$q_y = -\mathrm{sech}^2(\zeta), \quad x = \zeta - 1.5\tanh(\zeta), \quad q = \int^{\zeta} q_y y_\zeta \, \mathrm{d}\zeta, \tag{35}$$

then one can obtain a new kind of fission solitary wave solution for the expression U, which possesses apparently different evolutional property compared with the fusion phenomena in Figs. 1 – 3. From Figs. 4a – d, one can clearly see that the single bell-like semifoldon fissions into two bell-like semifoldons with different amplitudes. It is interesting to mention that these bell-like semifoldons run along different directions, i. e., the small one moves along the positive x-axis, and the big one travels along the negative x-axis. They are stable and do not undergo additional fission when running the program for longer periods.

5. Summary and Discussion

In conclusion, the general projective Riccati equation method is used to obtain variable separation solutions of the (2+1)-dimensional MDWW system. By further studying, we find that these variable separation solutions obtained by the PREM, which seem independent, actually depend on each other. Based on the quantity (27) and choosing suitable functions p and q, new types of fusion and fission phenomena among peakons, compactons, dromions and semifoldons are firstly investigated. Due to the experimental realization of soliton fusion and fission in many physical systems with some special conditions [2, 3], we think that the discussions here about soliton fusion and fission phenomena in higher-dimensional systems are significant and interesting. Clearly, there are still some pending issues to be further studied. How to quantify the notions of soliton fusion and fission phenomena? What is the general equation for the distribution of the energy and momentum for these soliton fusion and fission behaviors? What are the necessary and sufficient conditions for soliton fission and soliton fusion which have been pointed for the (1+1)-dimensional cases in [4 – 6]?

What we have obtained also verifies that the general projective Riccati equation method is quite useful to generate abundant localized excitations. Besides the (2+1)-dimensional MDWW system discussed in this paper, we can also obtain the variable separation solutions of the (2+1)-dimensional Korteweg-de Vries (KdV) equation, Boiti-Leon-Pempinelli (BLP) system, DLW system, BKK system, etc., by means of the PREM. For the limit of length, we do not list them here. In our future work, we will study how to generalize this method to the (1+1)-dimensional and (3+1)-dimensional nonlinear systems and to the differential-difference equations.

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